formulation of these concepts in terms of limits shows him both their real meaning (the understanding of which, by the way, is a thing worth while for every educated person), and also how he must proceed in order to calculate them. When he has seen these he has the basic ideas of the calculus.

In fact, before he suspects that he is dealing with anything so terrible, the student has differentiated numerous simple functions (polynomials in x and $1/x$) by the increment method, and has solved easy problems on instantaneous rates, differential corrections and maxima and minima. Of course these have to be selected with great care. After getting the formulas for differentiating positive and negative powers of x at sight he meets simple problems on motion, flexure of beams, and other rates, in which he needs to differentiate two or three times in succession; and other problems in which he needs to differentiate a power of a function.

The need of anti-differentiation soon arises in problems of motion, and also in calculating areas, work, volumes, fluid pressure, etc. No mention of definite integrals is made here; in fact, the practice of determining a constant of integration each time it arises both calls attention to its importance and helps the student later on to see the meaning of the term $-\phi(a)$ in $\int_a^b \phi'(x)dx$.

The student now meets some problems which he cannot solve completely because he does not know the anti-derivative, and some in which he cannot yet express the functional relation between the sides and angles of a triangle. He is reminded that he plotted numerous functions by using tabulated values, for which he knew no formula or mathematical expression; and is told that many sorts of expressions are used besides those he met in elementary algebra. Before he can make much headway it will be necessary to become familiar with various other functions besides simple powers, fractions, sums, etc.; and it will be helpful to consider, first, the way in which the sides and angles of a triangle are related.

The actual definition of the trigonometric functions and solutions of triangles is prefaced, however, by some little use of a protractor. With it the student solves roughly such triangles as arise in surveying or statics. This is nothing more mysterious than drawing a figure to scale and reading off the required parts. This construction work not only makes the student feel that he has a ready solution or check in any such problem, but also helps him to be clear about the application of trigonometry to these subjects. Moreover, before he has any trigonometric functions to think about, it is a good time for him to get the few principles of force-resultants, which he needs to know for the problems in statics. Any one can quickly master such problems as the graphical analysis of a simple framed structure composed of a few triangles. But, of course, we must lead him by natural steps from the parallelogram of forces to such analysis.

At this stage only four functions are introduced, and three-place values are used at first, checked by the protractor to emphasize their meaning. But larger tables are soon used which make the student feel the need of easier methods of calculating, in spite of various arithmetical devices and short cuts which are

shown him. In passing be it said that some oblique triangles are solved by dissecting them or by using the sine law and cosine law.

The work on logarithms is prefaced by use of the notation 2.417×10^{-8}, etc., so common in scientific work. This is both useful in itself and helpful in making clear the treatment of characteristics of logarithms. No rule for characteristics is needed if the student gets to thinking of every number as expressible in this "scientific notation." This saves some time and trouble, and illustrates one method by which much time is saved all through the course: viz., teaching *processes*, instead of rules which take time to learn and are likely to become mechanical. Another illustration of this is the matter of interpolation; in connection with the first graphical work the student learns to interpolate in any table by a method of proportional parts. Consequently he does not need later to be taught how to interpolate in trigonometric, logarithmic, or other numerical tables. A hint suffices to show him the short-cuts after he has had some practice in using the old general process. Of course, no base but 10 is even mentioned until all the actual computing and applications have been covered, which include formulas of physics, engineering, geometry, trigonometry, interest, etc. And it is interesting that students themselves prove the laws for any base, having seen their nature clearly while working with logarithms as exponents of 10.

The student is now ready to differentiate $\log v$ and e^v, to perform integrations involving these, and to solve problems on the "compound interest law." Here he learns also to differentiate products and quotients, which might have been taken up earlier, but at the risk of confusion. In the first calculus work the student had practically nothing in the way of formulas to think about; consequently his whole attention could be given to methods of applying the calculus, and to the meaning of the processes. Each time he learns to differentiate a new type of function, he makes applications which use all the old principles. Thus, through the year, there is frequent review of the underlying principles of rates, acceleration, calculation of areas, work, volumes, differential approximations, etc.

This is followed by further graphical work, but this time from the standpoint of coördinate geometry, while heretofore it has been based merely upon the representation of a function. Starting with the plotting of points, fixed or moving, the student sees how to study the motion of a point by its parametric equations, $x = f_1(t), y = f_2(t)$; finding the velocity and acceleration at any point, and the length of the path traveled. In particular, he deals with the motion of a projectile, both when its equations are given and when he must find them by integration. Next, by seeking a sure test as to whether a given point lies on a certain line or curve, the student sees the relation between a curve and its equation. He then derives the equations of various loci, proves analytically some simple theorems in the geometry of triangles and thus comes to see that algebraic methods may be systematically employed to study geometry. This is further emphasized by the discovery of some properties of conics (previously unknown to him) from their equations which he has set up. The idea of sliding a curve helps him to recognize the locus of any quadratic which lacks the xy term. Applica-

tions of the conics are pointed out here. This work is concluded by using ordinary graph paper to discover physical laws of the linear type, and logarithmic and semi-logarithmic paper to discover some other laws.

This is followed by work on the solution of equations, some of which, by the way, was included in the first graphical work. At this stage, the student is able to use differential approximations to advantage, and the idea of sliding a curve leads him easily to Horner's method. Of course, he also learns to find rational roots exactly by trial. The geometrical interpretation of simultaneous equations leads to some remarks on the fascinating topic of n-dimensional space.

The work on polar coördinates includes the plotting of fixed and moving points with special reference to circular motion, both uniform and non-uniform. In connection with this are studied the relation of an arc to its central angle, radians, estimates involving small angles, simple harmonic motion and the differentiation of the sine and cosine. The trigonometric functions are now defined for angles of any size, their graphs are drawn and their relations to acute angles are studied. Applications are made to alternating currents, parametric equations, etc.

The work on trigonometric analysis begins with the primary relations among the functions of a single angle (which the student has never yet heard of), and the uses of these relations, together with the double angle formulas, in differentiation and integration, including the method of rationalizing an algebraic integral by a trigonometric substitution. The usual formulas are developed for the direct logarithmic solution of oblique triangles and the addition formulas are proved and applied.

The student is now able to handle some of the fairly complicated integrals which arise in applications of the integral calculus. At the same time he learns to calculate volumes by double integrations in simple cases, and to draw plane sections of simple surfaces.

This is followed by some work on progressions and series with applications to interest and annuities, computation of logarithms and trigonometric functions, and integration by expansion into series. Some remarks on imaginary logarithms and Fourier's series are usually made at this point (to the student's interest, if not to his profound understanding!).

In conclusion, some work is usually done on elementary permutations and combinations and simple problems of chance. A brief introduction to the theory of probability lets the student use the method of least squares in the simplest case.

The course as given takes four hours a week through the year. The ground could be covered by lectures in less time; but with so wide a range of topics, it is important to allow time for considerable practice in class. Some time is saved by usually taking up a new topic before leaving the old,—that is, including review problems in nearly every assignment. The closeness with which successive topics are correlated and the frequency of contact with earlier topics make it possible for students to assimilate the material pretty well in spite of the rapidity

of their progress. Concise summaries in mimeographed form help greatly in fixing the grasp of the topics, especially as no text is available which covers any large part of the course from the same standpoint.

The treatment of topics is unconventional in that it keeps their practical uses steadily to the fore, and admits only topics, some of whose uses can be exhibited when the topic is studied. This gives students constant practice in applying their mathematics and in analyzing problems. It also creates considerable enthusiasm for the subject. Students see something of the meaning of mathematical work, and they do a very large amount of work very cheerfully—one may almost say eagerly. (Approximately one thousand exercises are worked during the year, many of which are fairly substantial problems.)

A word concerning the courses which follow. Naturally they must be modined somewhat. The second year is devoted to a systematic course in calculus, but several of the usual preliminaries, which were omitted from the first course, are treated incidentally as needed,—*e. g.*, elements of solid analytics before taking up multiple integration. Also, trigonometric analysis is reviewed in detail before differentiating trigonometric functions. Since the students are already familiar with the elements and general principles, they can be given an excellent grasp of the subject in this year's work. Besides covering a very full text-book treatment, they have time for some use of imaginaries in trigonometric reductions, for elementary problems on Fourier's series and calculus of variations, and for some practice in formulating as well as solving practical differential equations.

The more theoretical parts of college algebra and analytic geometry are postponed to the junior year when they can be dealt with adequately in connection with modern developments in algebra and geometry. Thus it is possible to devote the junior and senior years to mathematics of a fairly advanced type.

It seems to the writer that it would be perfectly feasible to give the freshman course outlined above in the fourth year of high schools and academies wherever there are strong teachers who know the subject well. Even then students would not be starting the study of these topics as young here as abroad. It is not hard to think of advantages which would result from their earlier introduction here.

LINKAGES.

By DICKSON H. LEAVENS, College of Yale in China, Changsha.

The subject of linkages is somewhat removed from the main lines of mathematics, but although it may not be of much importance in their development, it has considerable intrinsic interest. The problem that gave rise to the study of link-motions, that of making a point move in a straight line, is of significance theoretically in furnishing a method of drawing a straight line without begging the question as we do when we copy by means of a ruler a line already made.[1]

[1] See A. B. Kempe, *How to Draw a Straight Line*, London, Macmillan, 1877, which also gives a very clear and simple account of linkages in general. It is perhaps the best introduction to the subject, and its footnotes give references to the original papers up to its date.

This problem is also of practical importance in machinery, where it is necessary to produce rectilinear motion of certain parts. The history of the subject, too, is interesting, three or four great minds have contributed to it, and some remarkable results have been obtained. Finally, it sometimes furnishes a striking way of illustrating graphically results obtained analytically.

A plane linkage may be defined as a system of bars (or more generally, plane figures) pivoted together so that they move in the same plane (*i. e.*, in the case of material links, in close parallel planes). A *complete* linkage will have an even number of bars, so connected that there is only one degree of freedom of motion. A *linkwork* consists of an odd number of bars, pivoted at two points to a fixed plane.[1] If we consider the fixed plane as a link, we have a linkage, so that word is often used loosely instead of linkwork.

Leaving out of consideration the simple and familiar pantograph, invented early in the seventeenth century by C. Scheiner, we may say that the subject had its origin with James Watt. In his improvements on the steam-engine he found the need of guiding the piston rod in a straight line. This rectilinear motion is now effected by having the cross-head slide on smooth-planed guides, but this could not be done with sufficient mechanical perfection in his day, and some method with less friction was required.

In 1784 Watt patented his so-called "parallel motion." In a letter[2] to his son, written twenty-four years later, he says: "The idea originated in this manner. On finding double chains, or racks and sectors, very inconvenient for communicating the motion of the piston-rod to the angular motion of the working-beam, I set to work to see whether I could contrive some means of performing the same from motions turning upon centers, and after some time it occurred to me that AB and CD (Fig. 1),[3] being two equal radii revolving on the centers B and

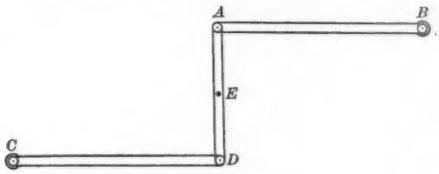

FIG. 1.

C, and connected together by a rod AD, and moving together through arches of certain lengths, the variations from the straight line would be nearly equal and opposite, and that the point E would describe a curve nearly straight, . . .

[1] For these definitions and for future references to Sylvester, see *On Recent Discoveries in the Mechanical Conversion of Motion*, in Sylvester, *Collected Works*, Cambridge Univ. Press, 1909, III, pp. 7–25. This paper is one of the most interesting on the subject.

[2] J. P. Muirhead, *The Life of James Watt*, New York, Appleton, 1859, p. 242.

[3] The properties of linkages are much more vividly brought out, both for individual study and for demonstration to a class, by working models than by mere diagrams. Such models are very easily made, after a little practice, with the links of moderately stiff cardboard, pivoted together with eyelets by means of an eyelet punch and set.

and from this the construction, afterward called the parallel motion, was derived. . . . Though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other invention I have ever made." The whole curve described by E is a figure eight sextic, with the two parts through the node deviating very slightly from a straight line.

Watt applied this to his engines, using the walking-beam for one of the fixed arms, and adding two more links in a pantograph arrangement so that a second point was made to move in a straight line parallel to that described by E. One of these points was attached to the piston-rod and the other to the air-pump rod.[1] It is probably this arrangement that was the origin of the name "parallel motion," which has been carried over, though obviously a misnomer, to the simpler form and to all other linkworks designed to give rectilinear motion to a point.

There does not seem to have been any scientific investigation of the subject during the first half of the nineteenth century. The first mathematician to take it up was Tchebychef (variously spelled as romanized in different countries), professor in the University of St. Petersburg. He became interested in it while giving a course in applied mechanics, and in 1852 took a trip[2] to western Europe, especially to visit factories, see different types of mechanism, and most of all to study the Watt parallel motion. In England he hunted up Watt's original machines,[3] wherever he could find them, in order to ascertain the lengths of the various parts. On his return, he studied the question of getting the closest possible approximation to rectilinear motion by linkwork. This led to an analytical problem which he treated skilfully and at great length, inventing new methods of analysis which are beyond the scope of this paper. His results enabled him to discuss the Watt movement, and to devise new ones, all of them only approximately straight line motions, but in every case accompanied by formulas showing the amount of deviation from a straight line as a function of the arbitrary constants of the linkwork. This deviation, by a proper choice of the constants, can be made very small, well within the limits of mechanical accuracy in the construction, so that for practical purposes they are quite as useful as if theoretically exact.

Tchebychef had considered the problem of obtaining exact rectilinear motion, and had finally come to the conclusion that it was impossible, although he did not succeed in proving this. In 1871, however, one of his own students, Lipkin, discovered a seven-bar linkwork which would describe an exactly straight line. The Russian government awarded him a prize, and then it transpired that identically the same device had been discovered by a French officer, Peaucellier,

[1] See the *Century Dictionary*, 1911, p. 4278, under "parallel" for picture of application to walking-beam.

[2] For his own interesting account of the trip, see *Œuvres de P. L. Tchebychef* [in French], St. Petersburg, 1899, Vol. II, pp. vii–xviii. The various papers giving his work on linkages and the functions associated with them will be found in the two volumes. A brief general survey of his life and work, and a special account of his results on linkages, are contained in *P. L. Tschebyschef und seine wissenschaftlichen Leistungen*, Wassilief and Delaunay, Teubner, 1900.

[3] A very interesting collection of original engines and models made by James Watt and his workmen may now be seen in the Science Division of the South Kensington Museum, London.

in 1864, and announced, but not described, in the *Nouvelles Annales*[1] of that year, without attracting much attention. Peaucellier's priority and Lipkin's independence were both fully established.

This seven-bar linkwork (Fig. 2) consists essentially of a six-bar linkage, called the Peaucellier cell, and an extra link. The cell is composed of a rhombus, P_1MP_2M', and two equal bars, OM, OM', pivoted to opposite vertices and pivoted

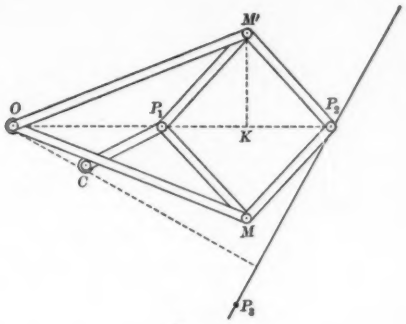

FIG. 2.

together at O. By symmetry the three points O, P_1, P_2 (respectively called the fulcrum, first pole, and second pole) are always collinear, and by simple geometry:

$$\overline{OP}_1 \cdot \overline{OP}_2 = (\overline{OK} - \overline{P_1K}) \cdot (\overline{OK} + \overline{P_1K}) = \overline{OK}^2 - \overline{P_1K}^2$$

$$= (\overline{OM}^2 - \overline{KM}^2) - (\overline{P_1M}^2 - \overline{KM}^2) = \overline{OM}^2 - \overline{P_1M}^2, \text{ a constant.}$$

That is, the product of the distances of the two poles from the fulcrum is constant. Thus, keeping O fixed, if P_1 describes any curve, P_2 will describe the inverse curve. It is a well-known fact, which may be shown by elementary geometry, that the inverse of a circle is in general a circle, but when the center of inversion lies on the original circle, the inverse is a straight line. Hence by constraining P_1 to move in a circle through O, which may be done by pivoting O to a fixed plane, and adding another link, CP_1 pivoted at C so that $CP_1 = OC$, we may make P_2 move in a straight line P_2P_3, perpendicular to OC. Or, by making $CP_1 \neq OC$, we may make P_2 move in a circular arc of any desired radius, by properly adjusting the constants.

This beautiful device aroused much interest in the study of linkages in general, and during the five years beginning 1874 more than one hundred[2] papers on the subject appeared in the mathematical journals. Various mathematical applications and generalizations of the Peaucellier apparatus were made. In mechanism, comparatively little use seems to have been made of it, for other methods of

[1] 2e série, III, p. 414.

[2] Liguine, *Liste de travaux sur les systèmes articulées, Bull. des Sci. Math.*, 2e série, VII, pp. 145–160, gives 150 titles up to 1882. For later bibliography see *Royal Society Catalogue of Scientific Papers, 1800–1900, Subject Index*, Vol. II, *Mechanics*, pp. 84–86.

getting rectilinear motion are more convenient in practice. Many special linkworks have been constructed to describe different curves, such as conics, cardioids, limaçons, lemniscates, cissoids, etc. The simplest conicograph contains seven bars. Linkages have been applied to complex variables and elliptic functions, and to realizing numerous algebraic transformations.[1]

The study of the curves described by a linkwork can most readily be made by the use of polar coördinates, which may later be transformed into Cartesian.[2] In general, a three-bar linkwork describes a sextic curve, and in general the curves described by any linkwork turn out to be algebraic. This and its converse were stated by Peaucellier[3] from general reasoning, which did not satisfy Sylvester, who outlined a proof. It remained for Kempe to prove explicitly: "A linkwork can be found to describe any given algebraic curve." There is no space here to outline his simple and elegant proof.[4] The method is much more complicated than most of the special linkworks devised for important curves, but it is of theoretical value. The opposite theorem is also true, that no transcendental curve can be described by a linkwork. These results definitely show us the large possibilities of linkworks, and also their limitations.

THE PROBABLE RANK IN A LARGE CLASS OF A STUDENT OF GIVEN RANK IN A SMALL CLASS.

By L. D. AMES, University of Missouri.

In a certain statistical investigation it was desirable to compare students in different classes on the basis of their ranks in their respective classes. To reduce the two rankings to a common unit we seek to know the probable rank in a very large class of a student who ranks kth in a class of n students.

For example, we may wish to know how a given class of students who have had a certain definite type of training compare in their subsequent work with the average of students with whom they may compete. We follow these particular students into various other classes and find, for example, that one of them ranks second in a class of nine, another ranks sixth in a class of ten, etc. We wish to assign a numerical value to these rankings. Does the second student in a class of four probably rank, other things being equal, on a par with the twenty-second student in a class of forty-four?

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[1] See Emch, *An Introduction to Projective Geometry and Its Applications*, Wiley, 1905, for a chapter on linkage transformations and references to original papers.

[2] For examples of the methods of attack, see F. Dingeldey, *Über die Erzeugung von Kurven vierter Ordnung durch Bewegungsmechanismen*, Teubner, 1885, Chapter III. Also Königs, *Leçons de cinématique*, Paris, 1897, Chap. XI.

[3] *Note sur une question de géometrie de compas, Nouvelles Annales*, 2e série, XII, p. 71, 1873. This, by the way, is Peaucellier's first published description of his invention.

[4] *On a General Method of Describing Curves of the nth Degree by a Linkwork. Proc. of London Math. Soc.*, 1876, VI, pp. 213–216. Also given in Königs, *op. cit.*, pp. 269–273.

To answer this question we suppose that a very large number of students are equally distributed along a line of given length. A class of n students is picked at random from this large number. The kth student of this class, counting from one end, is noted, and his distance from that end. This is repeated a large number of times and the distances averaged. What is the probable value of this average?

Clearly the second student in a class of three, and the fifth student in a class of nine stand at the middle of their respective classes, or at 1/2 on a unit scale. The result reached below agrees with this conclusion. In a class of three the first student would be ranked as 1/4, the second as 2/4, the third as 3/4. In general, if a scale of any convenient length is divided into equal parts one greater in number than the number of students in the class, and the students are ranked on the scale at the points of division in the order of their rank in the class, the end points of the scale not being used, then their distances from one end will represent their probable ranks in a very large class.

To avoid irrelevant assumptions we state the problem in abstract form. If n points be independently chosen at random on the interval from $x = 0$ to $x = 1$, find the probable distance from $x = 0$ to the kth point counting from that end. By the words "chosen at random" we mean that the probability that any specified point will be taken from any interval is the same as the probability that it will be taken from any equal interval.

Let us think of the points as chosen successively. The probability that the first point chosen will be chosen from the interval from x to $x + \Delta x$ is precisely Δx. The probability that the next $k - 1$ points chosen will be chosen from the interval from 0 to x is x^{k-1}. The probability that the last $n - k$ points chosen will be chosen from the interval from x to 1 is $(1 - x)^{n-k}$. The probability that all of these things will happen together is the product of their separate probabilities. But, as any one of the n points is equally likely to be the kth point from the origin, we must multiply this by n, and as for each one of these cases the number of different possibilities of selecting the points which fall in the interval from 0 to x is the number of combinations of $n - 1$ things taken $k - 1$ at a time, the probability that the kth point from the origin will lie in the interval from x to $x + \Delta x$ is the product of these probabilities, or

$$n \cdot {}_{n-1}C_{k-1}x^{k-1}(1 - x)^{n-k}\Delta x.$$

If now we multiply this probability by x, the distance from the origin to a point of the interval, add these terms for all the intervals, take the limit as Δx approaches zero and divide by the total probability which is 1, we have as the probable average distance from the origin to the kth point counting from the origin the definite integral

$$n \cdot {}_{n-1}C_{k-1}\int_{x=0}^{x=1} x^k(1 - x)^{n-k}dx.$$

Successive integration by parts gives the probable distance as $k/(n + 1)$.

If both classes are supposed finite, we may ask what is the probable rank, K, in a class of N students of the kth student in a class of n students. We have $K/(N+1) = k/(n+1)$, or $K = k(N+1)/(n+1)$. Thus, for example, the second student in a class of four, according to the above assumptions would stand at a distance of 2/5 from the origin, and would rank the same as the eighteenth student in a class of 44.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

The Development of the Arabic Numerals in Europe Exhibited in Sixty-four Tables. By G. F. HILL. Oxford, Clarendon Press, 1915.

In *Archæologia*, Vol. LXII, Mr. G. F. Hill, keeper of the department of medals and coins in the British Museum, published a series of fifty-one tables of Hindu-Arabic numerals as they appeared in Mss. and on monuments, coins, seals, medals, brasses, and paintings, as well as a few forms from printed works. This article has been of great value to all who are interested in the exact dating of mediæval manuscripts, for the work furnishes, where numerals are used, definite checks upon the time when a manuscript was written. Further and more particularly, the work is of interest to students of the history of science for it shows in graphical form the slow but sure progress of the Hindu arithmetic through Europe. Hearty welcome, then, is given to the present publication of the Oxford Press, in which appear not only the tables of the earlier work, but also additional tables of forms more recently located.

In general the forms included are only those which antedate 1500 A.D., for by that time the new arithmetic had become well-nigh universal. Parenthetically it is of interest to note, however, that as late as 1540, Köbel, in Germany, published an arithmetic wholly, except paging, in Roman numerals.

The earliest European forms are doubtless those found in the *Codex Vigilanus*, written 976 A.D. in the monastery of Albelda near Logrono in Spain. A second Spanish manuscript of about the same date, not described by Mr. Hill, also contains similar forms, and facsimiles. Both are to appear in the next issue of Professor John M. Burnam's *Palæographia iberica*. The numerals appear to have been known in Syria in 662 A.D., for a Syrian Bishop, Severus Sebokt, of a monastery on the Euphrates, refers in a work of that time to the "easy method of their, *i. e.*, the Hindus', calculations and of their computations which surpasses words. I mean that made with nine symbols," referring certainly to the Hindu system of numerals with the zero.

Interesting also is the fact that Mr. Hill finds that the forms which afford the best criteria for dating are those for 2, 4, and 7, while 5, "the most freakish of all figures," comes next. In probably the earliest translation of an Arabic treatise on the Hindu arithmetic, the *Algoritmi de numero Indorum* (published by Bon-

compagni in 1857), a translation made quite certainly in the early part of the twelfth century, the unknown writer states that there is some diversity among men in writing the figures for 5, 6, 7, and 8. This may even have been in Al-Khowarizmi's lost original treatise on arithmetic; no forms are found at this point in the unique Cambridge MS. of the translation. In passing we may note that the 2 written in form like a 7 is indicative of writing earlier than about 1,350 while a three which resembles a printed r (small r) is characteristic of the twelfth or early thirteenth century.

Mr. Hill has previously established a reputation in the field of numismatics, particularly Greek, and in Greek history. In the present treatise Mr. Hill has rendered a real service to the history of science by his patient and long-continued search for early appearances of the Hindu-Arabic numerals in Europe and by their publication in this convenient form.

The price of the book is $1.00; and it may be obtained from the Oxford University Press, 29 W. 32d St., New York City.

LOUIS C. KARPINSKI.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

445. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a \binom{cn-ca}{a}.$$

446. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Solve the equations

$$x^2(y-z) = l^2(m-n), \qquad y^2(z-x) = m^2(n-l), \qquad z^2(x-y) = n^2(l-m).$$

447. Proposed by ELIJAH SWIFT, University of Vermont.

(a) A method is sought of forming an equation such that the first k figures of some root shall be given numbers. For example, form a cubic equation such that one of its roots is 1.918 +.

(b) Of all the equations suggested in (a), determine that one for which the sum of the absolute values of the coefficients is least.

GEOMETRY.

476. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the locus of the middle points of a set of parallel chords intercepted between an hyperbola and its conjugate is $4b_1^2x^2y^2 - 4a_1^2y^4 = a_1^2b_1^4$.

Ashton's *Analytical Geometry*, p. 194, Prob. 34.

477. Proposed by N. P. PANDYA, Sojitra, India.

A village A_0 is equidistant from ten villages, $A_1, A_2, A_3, \cdots, A_{10}$. The distances, $A_1A_2, A_2A_3, \cdots, A_9A_{10}, A_{10}A_1$, are in arithmetic progression. A person, starting from A_0, has to go to four of these villages consecutively, and has then to return to A_0. What four villages should he select so that the total distance traveled by him may be a minimum?

478. Proposed by J. A. CAMPBELL, St. Johnsbury, Vt.

Construct a triangle having given the difference of the segments of the base formed by the foot of the perpendicular from the vertical angle, the difference of the base angles, and the sum of the three sides.

CALCULUS.

397. Proposed by C. N. SCHMALL, New York City.

On the radii vectores of one loop of the lemniscate, $\rho^2 = a^2 \cos 2\theta$, as diameters, circles are described, passing through the pole. Find the locus of their points of intersection and show that its area is twice that of the loop.

398. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$$

Proposed by EMMA GIBSON, Drury College.

Solve the differential equation,

$$(x^2y^3 + x^2y^2 + xy + 1)y + (x^2y^3 - x^2y^2 - xy + 1)\,x\frac{dy}{dx} = 0.$$

From Forsyth's *Differential Equations*, p. 48, Ex. 1, XXIV.

399. Proposed by B. J. BROWN, Victor, Colorado.

A cow is tethered by a perfectly smooth rope, a slip noose in the rope being thrown over a large square post. If the cow pulls the rope taut in the direction shown in the figure, at what angle will the rope leave the post?

From Granville's *Diff. and Int. Calculus*, p. 120, Prob. 55.

MECHANICS.

317. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the maximum area contained between the path of a projectile and the horizontal line is $\frac{v^2\sqrt{3}}{8g^2}$, where v is the velocity of projection.

318. Proposed by C. N. SCHMALL, New York City.

Given an inclined plane making an angle ϕ with the horizontal. A perfectly elastic ball is projected upward at an angle ψ with the inclined plane, so as to ascend it by bounds. Show that as the ball rebounds for the nth time, the angle of inclination of its path to the plane is

$$\tan^{-1}\left(\frac{\tan\psi}{1 - 2n\tan\phi\tan\psi}\right);$$

and if it rebounds vertically upward, then $\cot\psi = (2n+1)\tan\phi$.

NUMBER THEORY.

235. Proposed by W. D. CAIRNS, Oberlin College.

Prove that $n = 1$ is the only positive integer for which $n^4 + 4$ is a prime.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy + z = \square, \qquad yz + x = \square, \qquad \text{and} \qquad xz + y = \square.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

434. Proposed by S. A. JOFFE, New York City.

Express the "difference of zero" $\Delta^n 0^{n+1}$ in the form: $c_1(n+2)! - c_2(n+1)!$, where c_1 and c_2 are numerical coefficients independent of n.

SOLUTION BY THE PROPOSER.

From the theory of Finite Differences we know that

$$\Delta^n 0^i = n(\Delta^n 0^{i-1} + \Delta^{n-1} 0^{i-1}).$$

If we put here $i = n+1$, we obtain

$$\Delta^n 0^{n+1} = n(\Delta^n 0^n + \Delta^{n-1} 0^n),$$

which, after division by $n!$, gives

$$\frac{\Delta^n 0^{n+1}}{n!} = \frac{\Delta^n 0^n}{(n-1)!} + \frac{\Delta^{n-1} 0^n}{(n-1)!},$$

or, since $\Delta^n 0^n = n!$,

$$\frac{\Delta^n 0^{n+1}}{n!} - \frac{\Delta^{n-1} 0^n}{(n-1)!} = n. \tag{1}$$

Replacing n in equation (1) successively by $n-1, n-2, \cdots, 1$, we get the following $n-1$ similar equations:

$$\frac{\Delta^{n-1} 0^n}{(n-1)!} - \frac{\Delta^{n-2} 0^{n-1}}{(n-2)!} = n-1; \quad \frac{\Delta^{n-2} 0^{n-1}}{(n-2)!} - \frac{\Delta^{n-3} 0^{n-2}}{(n-3)!} = n-2; \quad \cdots \frac{\Delta^2 0^2}{2} - \frac{\Delta 0^2}{1} = 2; \quad \frac{\Delta 0^2}{1} = 1.$$

Adding together the preceding n equations, including (1), we obtain, after cancellation,

$$\frac{\Delta^n 0^{n+1}}{n!} = n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2},$$

which may be written in this form:

$$\Delta^n 0^{n+1} = \frac{n}{2}(n+1)! \tag{2}$$

Finally, since $\frac{n}{2} = \frac{1}{2}(n+2) - 1$, equation (2) may be expressed as follows:

$$\Delta^n 0^{n+1} = \frac{1}{2}(n+2)! - (n+1)! \tag{3}$$

which is the form desired.

435. Proposed by C. N. SCHMALL, New York City.

Show that $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \cdots = 1$, where e is the Naperian base of logarithms.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

This equation results from substituting $(e-1)$ for x in the series,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

This, however, is not a permissible substitution, since the logarithmic series

converges only when $x \gtreqless 1$. Moreover, the given series is not convergent, since

$$\operatorname{Lt}_{n=\infty} \frac{(e-1)^n}{n} \neq 0,$$

$e - 1$ being greater than unity.

Also solved by HORACE OLSON, C. E. FLANAGAN, J. W. CLAWSON, and H. S. UHLER.

GEOMETRY.

449. Proposed by H. E. TREFETHEN, Colby College.

Find a tetrahedron with the face angles at one vertex in arithmetical progression and its six edges expressed in positive integers.

SOLUTION BY THE PROPOSER.

Let the angles be $A + B$, A, $A - B$; the lateral edges x, y, z; the base edges a, b, c; so that $a^2 = x^2 + y^2 - 2xy \cos(A + B)$, $b^2 = x^2 + z^2 - 2xz \cos A$, $c^2 = y^2 + z^2 - 2yz \cos(A - B)$. If, for brevity, we put $1 + \cos(A + B) = P/2$, $1 + \cos A = Q/2$, and $1 + \cos(A - B) = R/2$, we may write

$$a^2 = (x+y)^2 - Pxy = (x+y-p)^2 \text{ or } Pxy - 2py = 2px - p^2, \tag{1}$$

$$b^2 = (x+z)^2 - Qxz = (x+z-q)^2 \quad Qxz - 2qz = 2qx - q^2, \tag{2}$$

$$c^2 = (y+z)^2 - Ryz = (y+z-r)^2 \quad Ryz - 2ry = 2rz - r^2. \tag{3}$$

Eliminating y and z from (1), (2), (3), and arranging we have

$$(2px - p^2)[q(4r - Rq) + 2(Rq - Qr)x] = r(Px - 2p)[2q(r - q) + (4q - Qr)x]. \tag{4}$$

If we put the coefficient of $x^2 = 0$, then

$$p = \frac{Pr(4q - Qr)}{4(Rq - Qr)} \tag{5}$$

and also

$$2x = \frac{p^2q(4r - Rq) + 4pqr(q - r)}{p(8qr - Qr^2 - Rq^2) - p^2(Rq - Qr) + Pqr(q - r)}. \tag{6}$$

We may now assign values to A and B and thus determine P, Q, R. If values are assigned to q and r also, p is defined by (5), and then x may be found from (6), y and z from (1), (2), (3) and finally a, b, c also.

Thus if $A = 90°$ and $B =$ arc-sin $1/3$, then $P = 4/3$, $Q = 2$, $R = 8/3$; and if also $q = r = 1$, then $p = 1$, $x = 1/4$, $y = 3/10$, $z = 1/3$. Reducing the values of x, y, z to a common denominator and rejecting it, we have in integers $x = 15$, $y = 18$, $z = 20$, and consequently $a = 27$, $b = 25$, $c = 22$. Or since the equations are symmetrical we may use the reciprocals $x = 4$, $y = 10/3$, $z = 3$, whence in integers $x = 12$, $y = 10$, $z = 9$, and then $a = 18$, $b = 15$, $c = 11$. Again if $\sin B = 1/2$, the angles are 120°, 90°, and 60°; $x = 9$, $y = 15$, $z = 40$; $a = 21$, $b = 41$, $c = 35$. If $\sin B = 2/3$, we find $x = 1{,}092$, $y = 416$, $z = 81$; $a = 2{,}804$, $b = 2{,}185$, $c = 367$.

By varying q, r and the $\sin B$, other sets of numbers may be found ad libitum.

464. Proposed by FRANK R. MORRIS, Glendale, Calif.

The sum of the hypotenuse and one side of a right triangle is 100 feet. A point on the hypotenuse is 10 feet from each of the sides. Find the length of the hypotenuse correct to the third decimal place.

SOLUTION BY W. W. BURTON, Mercer University, Macon, Ga.

Let $BC = x$. Then $AB = 100 - x$ and $BD = x - 10$. The right triangles ACB and PDB are similar. (Their sides are respectively parallel.) Therefore

$$AB : BC = PB : BD \text{ or } (100 - x) : x = PB : (x - 10),$$

and

$$PB = \frac{(x-10)(100-x)}{x}.$$

Again

$$\overline{PB}^2 = \overline{PD}^2 + \overline{BD}^2 \text{ or } \overline{PB}^2 = 10^2 + (x-10)^2 \text{ or } x^2 - 20x + 200.$$

Therefore

$$PB = \sqrt{x^2 - 20x + 200},$$

and

$$\frac{(x-10)(100-x)}{x} = \sqrt{x^2 - 20x + 200}.$$

Reducing, we get the cubic,

$$2x^3 - 139x^2 + 2200x - 10000 = 0.$$

Applying Descartes' rule of signs we find that all of the roots of the above cubic are positive. Applying Sturm's theorem we find that the roots are situated as follows: one root between 9 and 10; one root between 11 and 12; and one root between 49 and 50.

The root between 9 and 10 must be discarded, as it is impossible in our problem. Applying Horner's method for incommensurable roots, we find the roots to be 11.282 and 49.212 correct to three decimal places. Therefore BC = 11.282 or 49.212. Hence, the hypotenuse can be *88.718* ft. or *50.788* ft.

Also solved by HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, M. HELEN KELLEY, C. E. FLANAGAN, W. E. WHITFORD, G. H. HARTWELL, and NATHAN ALTSHILLER.

465. Proposed by ROGER A. JOHNSON, Western Reserve University.

Let C be a fixed circle, A a point outside it. Let AT and AT' be the tangents from A to the circle, touching the latter at T and T'. Let two secants be drawn through A, cutting the circle at P, Q and R, S respectively. Let PR and QS meet at X, PS and QR meet at Y. Prove by elementary methods that for all positions of the secants, X and Y lie on the line TT'.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

I. $PQSR$ is a quadrangle, having X, Y, A for its diagonal points. Hence $X(ARYS)$ is a harmonic pencil. Then, if XY cuts ARS at Z, $(ARZS)$ is a harmonic range.

Now TT' is the polar of A with respect to the circle whose center is C. Hence, if TT' cuts ARS at Z', $(ARZ'S)$ is a harmonic range.

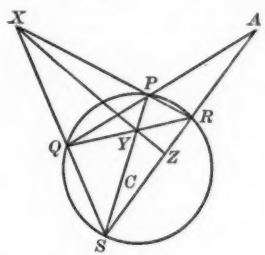

FIG. 1.

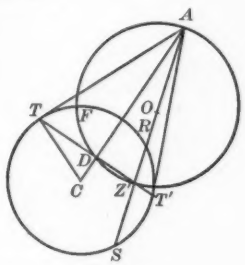

FIG. 2.

Therefore, the points Z and Z' coincide and XY and TT' cut the line ARS at the same point. Similarly, it can be shown that XY and TT' cut the line APQ at the same point. Hence the lines XY and TT' coincide.

[The exercise follows at once from the theorem: If a system of conics circumscribe a given quadrangle, the diagonal point triangle is a self-conjugate triangle w. r. t. each conic of the system. (Durell, Plane Geometry for Advanced Students, Part ii, p. 110.)]

If the above proof is not considered "elementary," a more detailed proof, avoiding the use of the terms "harmonic range" and "polar" is as follows:

II. Extend XY to meet ARS at Z.

Then, since XZ, SP, RQ are concurrent at Y, by Ceva's Theorem,

$$\frac{SZ}{ZR} \cdot \frac{RP}{PX} \cdot \frac{XQ}{QS} = 1.$$

Again, since APQ is a transversal cutting the sides of the triangle XSR, by Menelaus' Theorem,

$$\frac{SA}{AR} \cdot \frac{RP}{PX} \cdot \frac{XQ}{QS} = -1.$$

Comparing these results, we see that

$$\frac{SZ}{ZR} = -\frac{SA}{AR}.$$

Again, let TT' cut SR at Z'. From A draw AD perpendicular to TT'. On AZ' as diameter draw a circle. This circle passes through D. Bisect AZ' at O. Let one of the points of intersection of the two circles be F.

Now, $\triangle ACT$, TCD are similar. Hence $CT^2 = CD \cdot CA$, *i. e.*, $CF^2 = CD \cdot CA$. Therefore, CF is a tangent to circle O. Hence, $\angle CFO$ is a right angle. Hence, OF is a tangent to circle C. Hence, $OF^2 = OR \cdot OS$, *i. e.*, $OZ'^2 = OR \cdot OS$, *i. e.*,

$$\frac{OS}{OZ'} = \frac{OZ'}{OR},$$

i. e.,

$$\frac{OS - OZ'}{OS + OZ'} = \frac{OZ' - OR}{OZ' + OR},$$

i. e.,

$$\frac{Z'S}{AS} = \frac{RZ'}{AR}.$$

Hence,

$$\frac{SZ'}{Z'R} = -\frac{SA}{AR}.$$

Hence, the points Z and Z' coincide. Therefore, the lines XY and TT' intersect ARS at the same point. Similarly it can be shown that XY and TT' intersect APQ at the same point. Hence XY and TT' coincide.

Also solved by N. P. PANDYA.

CALCULUS.

378. Proposed by ELBERT H. CLARKE, Purdue University.

The area of the curved surface generated by the revolution about OX of the portion of the curve $y = x^n$ which extends from the origin to the point (1, 1) is given by the formula

$$A = 2\pi \int_0^1 x^n \sqrt{1 + n^2 x^{2n-2}} \cdot dx.$$

Our geometric intuition would tell us that the limit of this area as n becomes infinite is π. Give a strict analytic proof that

$$\lim_{n=\infty} \int_0^1 x^n \sqrt{1 + n^2 x^{2n-2}} \cdot dx = \tfrac{1}{2}.$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We give the proof by writing the above integral as the sum of two and showing that the limit of one is zero, of the other, $\frac{1}{2}$. Let k be any value between 0 and 1, *e. g.*, .9. Then

$$\int_0^1 x^n \sqrt{1 + n^2 x^{2n-2}} \cdot dx = \int_0^k x^n \sqrt{1 + n^2 x^{2n-2}} \cdot dx + \int_k^1 x^n \sqrt{1 + n^2 x^{2n-2}} \cdot dx.$$

But the integrand of the first of these two integrals is less than (or equal to) $k^n\sqrt{1+n^2k^{n-2}}$ and since the limit of this is 0, the limit of the integral is zero also. To handle the second integral, write the integrand as

$$nx^{2n-1}\sqrt{1+\frac{1}{n^2x^{2n-2}}}.$$

We may now develop in series and show that the limit of every term after the first is zero, or we may proceed as follows:

$$\int_k^1 nx^{2n-1}\sqrt{1+\frac{1}{n^2x^{2n-2}}}\,dx = \int_k^1 nx^{2n-1}\left\{1+\sqrt{1+\frac{1}{n^2x^{2n-2}}}-1\right\}dx$$

$$= \int_k^1 nx^{2n-1}\cdot dx + \int_k^1 n\cdot x^{2n-1}\cdot\frac{\dfrac{1}{n^2x^{2n-2}}}{\sqrt{1+\dfrac{1}{n^2x^{2n-2}}}+1}\cdot dx,$$

by rationalizing the numerator. The limit of the first integral is $\frac{1}{2}$ and since the integrand in the second may be made as small as we please, the limit of the second integral is zero.

Also solved by J. A. CAPARO and the PROPOSER.

379. Proposed by C. N. SCHMALL, New York City.

Express the equation of the folium, $x^3+y^3=3axy$, in parametric form and find the area of the loop.

(From E. B. Wilson's *Advanced Calculus*, p. 296, ex. 5.)

SOLUTION BY E. B. WILSON, Mass. Institute of Technology.

Let $y=mx$, then

$$x=\frac{3am}{1+m^3}, \qquad y=\frac{3am^2}{1+m^3};$$

the loop being described by values of m from 0 to ∞. By the formulas for area as a curvilinear integral

$$A=-\int_{m=0}^{\infty} y\,dx=-\int_0^{\infty}9a^2\frac{m^2(1-2m^3)dm}{(1+m^3)^3}=-\int_0^{\infty}3a^2\frac{1-2u}{(1+u)^3}\,du,$$

where $u=m^3$. Then

$$A=-3a^2\left[\frac{2}{1+u}-\frac{3}{2}\frac{1}{(1+u)^2}\right]_0^{\infty}=\frac{3}{2}a^2.$$

Also solved by ELIJAH SWIFT, C. E. HORNE, WILSON L. MISER, W. C. EELLS, HORACE OLSON, J. A. CAPARO, H. L. AGARD, L. G. WELD, and the PROPOSER.

MECHANICS.

297. Proposed by C. N. SCHMALL, New York City.

A shrapnel shell strikes the ground and then explodes, dispersing its fragments in all directions with a common velocity v. If a be the area of the ground covered by the fragments, and if the dimensions of the shell be neglected, show that $a=\pi v^4/g^2$.

SOLUTION BY HORACE OLSON, Chicago, Illinois.

According to the laws of physics, the range of a projectile on the horizontal plane from which it is thrown is $(2v^2\sin\theta\cos\theta)/g$ or $(v^2\sin 2\theta)/g$, θ being the inclination with the horizontal of the line of projection. This range has a maximum value, v^2/g, when θ is 45°. Therefore the area of the ground covered by the fragments is $\pi v^4/g^2$, the area of a circle of radius v^2/g.

Also solved by A. M. HARDING, J. L. RILEY, and P. PEÑALVER.

298. Proposed by C. N. SCHMALL, New York City.

A person desires to throw a stone so as to strike the greatest possible blow at a point in a smooth vertical wall at a height h above the ground. If his strength is sufficient to throw the stone vertically upwards to a height $2h$, show that he must stand at a distance $2h$ from the wall. (The resistance of the air and the height of the hand are not taken into account.)

SOLUTION BY PAUL CAPRON, United States Naval Academy.

As the wall is smooth, the force of the blow is proportional to the square of the horizontal velocity of the stone, which in turn is the product of the given initial speed of the stone, v_0, and $\cos\alpha$, if α is the angle of elevation at the start. The problem is therefore to make $\cos\alpha$ a maximum, or to make α a minimum. $(0 < \alpha < \pi/2.)$ Let the required distance from the wall be z; then

$$z = v_0 \cos\alpha t, \qquad h = v_0 \sin\alpha t - \tfrac{1}{2}gt^2;$$

whence

$$2hv_0^2 = 2v_0^2 z \tan\alpha - gz^2 \sec^2\alpha.$$

Differentiating,

$$d\alpha(v_0^2 z \sec^2\alpha - gz^2 \sec^2\alpha \tan\alpha) = dz(gz \sec^2\alpha - v_0^2 \tan\alpha).$$

$$\frac{d\alpha}{dz} = 0$$

if

$$gz \sec^2\alpha - v_0^2 \tan\alpha = 0;$$

i. e., if

$$z = \frac{v_0^2}{g} \sin\alpha \cos\alpha.$$

Then

$$2hv_0^2 = \frac{2v_0^4}{g} \sin^2\alpha - \frac{v_0^4}{g} \sin^2\alpha = \frac{v_0^4}{g} \sin^2\alpha;$$

i. e.,

$$\sin^2\alpha = \frac{2gh}{v_0^2},$$

and therefore,

$$z = \sqrt{\frac{2h}{g}(v_0^2 - 2gh)}.$$

According to the given conditions,

$$v_0^2 = 2g \times 2h = 4gh,$$

so that $\alpha = 45°$, $z = 2$h.

The stone is at the highest point of its trajectory when it strikes, whatever the value of v_0.

We may also solve the problem as follows:

If the stone strikes the wall with a velocity v_1, at an angle β with the horizontal, the energy used up by the blow will be $\frac{1}{2}mv_1^2 \cos^2\beta$, where $v_1^2 = v_0^2 - 2gh$, and $v_1 \cos\beta = v_0 \cos\alpha$. v_0 and h being given, it is necessary for the greatest effect that $\beta = 0$. Then $v_1 = v_0 \cos\alpha$, $v_0^2 \sin^2\alpha = 2gh$, $\sin^2\alpha = 2gh/v_0^2$, and since $2hv_0^2 = 2v_0^2 z \tan\alpha - gz^2 \sec^2\alpha$,

$$2h = 2z\sqrt{\frac{2gh}{v_0^2 - 2gh}} - gz^2 \frac{v_0^2}{v_0^2 - 2gh}, \qquad \text{or} \qquad z^2 - 2\sqrt{\frac{2h}{g}(v_0^2 - 2gh)} \cdot z + \frac{2h}{g}(v_0^2 - 2gh) = 0;$$

i. e.,

$$z = \sqrt{\frac{2h}{g}(v_0^2 - 2gh)},$$

as before.

Also solved by MARCUS SKARSTEDT.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

27. A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytical geometry and differential and integral calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours offered?

REPLY BY W. A. HARSHBARGER, Washburn College, Topeka, Kansas.

This subject seems at first glance to be comparatively simple and to admit of a reasonably definite answer. Every teacher of college mathematics doubtless has a pretty clear idea of what subjects a student majoring in pure mathematics should pursue in the last two years of his undergraduate course, and it is quite likely that these foot up to twice twelve hours. The real problem is one of elimination.

Introduction. Let us suppose that the student enters with the ordinary high-school preparation and the ordinary development in mathematical reasoning. The freshman year is devoted to trigonometry and so-called college algebra, three hours being devoted to each subject. The classes are large and made up of students of various grades of ability from superior to microscopic. Also they contain, side by side, students who will not pursue the subject beyond the freshman year, students who expect to take engineering work, and the few, comparatively, who will specialize in pure mathematics. Also, unless my experience is exceptional, there will be various degrees of preparation in the fundamental operations of the subject, from good to that which is really worse than none at all. Consequently the instruction in the freshman year, particularly in the algebra, must, to a considerable extent, be given to strengthening and correcting the preparation. For this reason but little college algebra can be given in the freshman year. A brief, but thorough review of the fundamental operations, surds, exponents, etc.; then a thorough treatment of the quadratic in one and two unknowns, with as full a consideration of the graph as time will permit, will occupy half the term, or probably a little less. The more advanced topics, complex numbers, permutations and combinations, determinants, theory of equations, etc., thus receive somewhat scant treatment.

In the sophomore year I suppose the student to take five hours each of analytic geometry and calculus. Here again the classes are large and of a mixed character. The work in analytic geometry will cover thoroughly the straight line, circle and conic sections, with a brief course in the geometry of three dimensions, always with the object of teaching the analytical method, rather than of teaching facts about the conics themselves.

In the calculus I presuppose a considerable facility in the processes of differentiation and integration, with simple applications and a great many problems. Infinite series, expansion by Taylor's and Maclaurin's theorems, and the definite integral may be supposed to receive careful consideration, but little will be done with partial differentiation.

We have now conducted the class up to the point contemplated by the question and probably have a comparatively small number in line for the next twelve hours in pure mathematics. As I understand the question, the purpose is to select for the next twelve hours those subjects that will best fit the student for graduate work in pure mathematics. An inventory of his work up to this time will surely show a deficiency in algebra. Hence I would have him devote the first three hours to a somewhat advanced course in algebra.

Algebra. The aim of this course is not to teach the higher algebra as that term is generally understood, but to carry the ordinary processes to a stage where their power will be appreciated; to give facility in execution; in short, to make algebra a more powerful tool in future work. My own practice is to found this course on Chrystal's *Algebra* and make numerous references to other works. It should begin at the beginning of the subject, and give a solid foundation in the fundamental laws and processes of algebra, perhaps the most difficult part of the course for the average student. A topical outline of the course is hardly necessary here, as it will be varied somewhat to meet the needs of different classes. It may well be extended to include the most essential parts of theory of equations, and the general theory of integral functions. As the object is to give facility of execution, quite as much as to teach the theory of the subject, a considerable list of select problems should be assigned, and carefully solved.

For the next three hours I would choose the subject geometry.

Geometry. This course should probably begin with a short review of the elementary course in analytic geometry, treated in a more scientific manner. From this we proceed to homogeneous point and line coördinates and the principle of duality. Here a short account of the general treatment given in courses in geometry of position, where point, line and plane, as fundamental elements lead to a multiplicity of dualistic theorems, will widen the student's horizon, and possibly awaken a desire for a later course in projective geometry. Dropping back to the point and line, the theorems on the complete quadrilateral and harmonic division are followed by the more general theorems on the descriptive properties of conics by Pascal, Chasles and others. The study of the metric properties brings in the line at infinity and the circular points at infinity. A study of projection, cross ratio and involution will complete a course sufficiently difficult for all but the best grounded classes in the subject.

For the next three hours, that is, the first term of the senior year, I would have a course in advanced calculus.

Advanced Calculus. This should begin with a brief review of the elementary course, and take up the subjects in differential calculus that were omitted or taken too briefly in the first course. Here partial differentiation should receive careful treatment. The work in infinite series, Taylor's theorem, etc., given in the first course may be considerably extended. But at least two thirds of the time should be devoted to the integral calculus. The definite integral may here be treated in a more advanced way than was possible in the first course. The student should now receive careful, though necessarily brief instruction in such

subjects as the gamma functions, elliptic integrals, line, surface and space integrals, and his knowledge of the applications of the definite integral should be considerably extended. It has been my custom to found the integral part of this course on Volume 2 of Byerly's *Calculus*, and to make liberal references to other works for both theory and application.

For the last three hours I find it difficult to decide on a subject. There are many subjects entirely suitable that would add variety to the somewhat restricted schedule given above. In many ways a course in analytic mechanics would be ideal at this time. However, keeping the graduate school in mind as the goal toward which the student is working, and recalling Dr. Bolza's rather insistent claim that a student should have two years of calculus for admission to graduate work, I venture to suggest a course in differential equations.

Differential Equations. This can be made as difficult as the ability of the class will permit, and furnishes a splendid review in the processes of integration. No outline of this need be suggested, for it will of necessity be kept quite elementary. However it should be given as many points of contact with other subjects as time will permit.

This schedule, three hours of algebra, three hours of geometry, and six hours of calculus, will probably seem to lack variety. I have kept the graduate school in mind and to the best of my ability selected subjects accordingly. For those students whose school career must stop with the bachelor's degree, or who pursue other lines of study later, a different selection may be advisable.

Conclusion. While it is not directly pertinent to the question, I venture to suggest that some students enter college prepared to begin analytic geometry, and are thus in line for eighteen hours instead of twelve. This is by no means uncommon in the college with which I am connected, owing to the good work of the mathematics department of the Topeka high school. Others will frequently desire to carry two courses simultaneously. Hence there is considerable opportunity to introduce variety where the teaching force will permit. Also a number of two hour courses can be offered, considerably widening the scope of the above schedule. In this way such subjects as history of mathematics, theory of determinants, geometry of position, geometry of three dimensions, and many others may be introduced, and the student given a much wider view without subtracting from his foundation work.

NOTES AND NEWS.

The problem department of SCHOOL SCIENCE AND MATHEMATICS is now in charge of DR. J. O. HASSLER, of the Englewood High School, Chicago.

MR. CARL GARLOUGH has been appointed to an instructorship in mathematics in Wheaton College, Wheaton, Ill.

PROFESSOR FLORIAN CAJORI has an interesting article in *Bibliotheca Mathematica* for May, 1915, on "An integration antedating the integral calculus."

DR. U. G. MITCHELL gave an address recently before the teachers of Marion county, Kansas, on "A new point of view for text-books and teachers."

The American Mathematical Society met in regular session at Columbia University, New York City, on October thirtieth; the southwestern section met at Washington University, St. Louis, on November twenty-seventh.

"Number and the quadratic" is the title of a paper by PROFESSOR RICHARD MORRIS, of Rutgers College, appearing in the November number of *School Science and Mathematics*.

"The need of supervision in college teaching" is the title of an article in *School and Society* for October 9, 1915, which is worthy of consideration by all college teachers, not least by teachers of mathematics.

The University of Washington Publications in mathematical and physical sciences have been inaugurated with an extensive presentation of "An arithmetical theory of certain numerical functions" by ERIC T. BELL.

In *Science* for July 23 PROFESSOR W. H. ROEVER gives a comparative and critical *résumé* of the treatments of the meridional deviation of a falling body by himself, by PROFESSOR F. R. Moulton, and by DR. R. S. Woodward, already noted in early numbers of the MONTHLY.

An inquiry made several years ago in *L'Intermédiaire* by MR. E. B. ESCOTT for a complete study of the equation $aX^4 + bY^4 = cZ^2$ brought forth a very extended article by A. GÉRARDIN in the July number of that journal, in which he solves the problem by three different methods and supplements this by a bibliography.

PROFESSOR G. A. MILLER writes on "A few classic unknowns in mathematics" in the October number of *The Scientific Monthly*. In this article he lists several well known questions in the theory of numbers which are as yet unsolved but the nature of which can be understood even by persons quite limited in their knowledge of mathematics.

FRANKLIN, MACNUTT and CHARLES have published a "supplement" which is to take the place of the first forty-one pages of the 1913 edition of their text in calculus. The principal change consists in giving a more rigorous introduction for the derivative and differential notation. A list of corrections to the earlier edition is also given.

DR. G. J. KOLLEN, president emeritus of Hope College, Holland, Michigan,

died on September 5. For fifty years he had been in educational work, being at various times professor of mathematics, natural philosophy and applied mathematics. He was a graduate of Hope College, and held the honorary degree of doctor of laws from Rutgers College.

"The Human Significance of Mathematics" is the title of an address given by PROFESSOR C. J. KEYSER at Berkeley, Calif., August 3, 1915, at a joint meeting of the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science. The address is printed in *Science* for November 22, 1915. It is well worth the attention of every teacher of mathematics.

The Macmillan Company announces the publication of a manual of "Constructive Geometry" by PROFESSOR E. R. HEDRICK, a "Grammar School Arithmetic" by PROFESSOR F. CAJORI, and an "Elementary Algebra" by PROFESSOR CAJORI and LETITIA R. ODELL, the latter of the North Side High School of Denver.

In the October number of the *School Review* MR. RALEIGH SCHORLING, of the University of Chicago High School, reports the results of a survey made for the purpose of investigating the general problem of individual differences in the teaching of secondary school mathematics, as treated in the various institutions which offer practice-teaching courses. The article concerns more particularly the methods for dealing with the slower students.

The March–April number of *Rendiconti del Circolo Matematico di Palermo* contains an unusually large proportion of articles by Americans. These are "Aggregates of minors of persymmetric determinants" by PROFESSOR W. H. METZLER and L. H. RICE, "A certain class of transcendental curves" by DR. FLORENCE E. ALLEN, "Conjugate systems with equal tangential invariants and the transformation of Moutard" by PROFESSOR L. P. EISENHART, and a "Note on trigonometric interpolation" by DR. DUNHAM JACKSON.

Upon the invitation of the American Society of Mechanical Engineers, a joint committee representing seventeen scientific societies, including the American Mathematical Society, has been working on the question of "Standards for graphic representation." A brief preliminary report has been published for the purpose of soliciting criticisms and suggestions. Copies may be obtained from the American Society of Mechanical Engineers, 29 West 39th St., New York City. The price is ten cents. The representative of the American Mathematical Society is PROFESSOR H. E. HAWKES, of Columbia University.

The Carnegie Endowment for International Peace has just issued a pamphlet entitled "Problems about war for classes in arithmetic." This was prepared by Professor DAVID EUGENE SMITH, and will be sent to any teacher who requests a

copy. Letters should be sent to the Endowment, 407 West 117th Street, New York City. The purpose of the pamphlet is to bring before pupils at an impressionable age some idea of their responsibility for a wiser use of money on the part of the whole world than that which looks for a model to the present vast expenditures for armaments.

At Rutgers College a Mathematical Club has been formed among the undergraduates, and during the past year the following papers were presented:

"The cubic and the quartic," MR. EDWIN FLORANCE; "The fundamental theorem of algebra," MR. LOUIS B. GITTLEMAN; "Elementary theorems on the roots of an equation," MR. J. H. HUNTINGTON; "The fourth dimension," MR. JAMES B. SCARR; "The elements of life insurance," PROFESSOR STANLEY E. BRASEFIELD; "Arithmetical processes and the number field," DR. RICHARD MORRIS; "The solution of certain trigonometric equations," MR. HAROLD I. FAWCETT; "Some trigonometric series," MR. CLIFFORD P. OSBORNE; "Trilinear coördinates," DR. JOHN A. INGHAM. The MONTHLY comes to the College library and is read by the mathematical students and others.

The fifty-third annual meeting of the Minnesota Educational Association was held at Minneapolis, October 27–30, 1915. The mathematics section of the secondary department held two sessions, at one of which MR. W. D. REEVE, the newly appointed head of the department in the University of Minnesota high school, discussed a list of helpful books and journals for teachers of mathematics. Also, at the section for college teachers of education MR. REEVE gave a paper on a methods course for training high-school teachers of mathematics. At each of these sessions, PROFESSOR H. E. SLAUGHT, being called from the floor, laid emphasis upon individual stimulus for high-school teachers above their daily routine and upon the need of some systematic attention to the preparation of college instructors in mathematics with respect to the principles of good teaching.

The *Mathematics Teacher* for September, 1915, contains an extensive report of a committee on bibliography of which EUGENE R. SMITH is chairman. The list of books given covers many things which might be of interest to a teacher of secondary mathematics and which cannot be found in the "Bibliography of the teaching of mathematics" published by the U. S. Bureau of Education as Bulletin No. 503. In particular, it contains lists of text-books and exercise books in algebra, geometry, and trigonometry; publications of state departments, and of the U. S. Bureau of Education, relating to mathematics in the secondary schools; and association reports and college publications on the teaching of algebra, geometry, and trigonometry; covering in all twenty-two closely printed pages. This bibliography should be of great service to all teachers of mathematics.

According to the annual report in *Science* for October 22, 1915, on Doctorates in American Universities, there were 23 doctorates in mathematics conferred

during 1915. The average number for the past ten years is twelve per year, and the total to date is 297. The departments conferring more than ten degrees in 1915 are, in order, chemistry 85, botany 40, zoology 32, physics 31, geology 26, mathematics 23 and psychology 22. The universities conferring two or more degrees in mathematics in 1915 are, in order, Chicago 7, Harvard 3, Columbia, Cornell, Pennsylvania, and Yale each 2. Illinois, Johns Hopkins, Missouri, and Princeton each conferred one doctorate in mathematics. The total number of doctorates in the sciences for 1915 was 309 and the grand total in all branches was 556. The number conferred during the past eighteen years is 6,320.

It is altogether in line with the purposes of the MONTHLY to call attention to notable texts in higher mathematics as they appear. In this connection we may mention among the earlier numbers of the Edinburgh Mathematical Tracts, published by G. Bell and Sons, under the general editorship of PROFESSOR E. T. WHITTAKER, the following numbers: "A course in interpolation and numerical integration for the mathematical laboratory" by DAVID GIBB; "Relativity" by A. W. CONWAY; "A course in Fourier's analysis and periodogram analysis for the mathematical laboratory" by G. A. CARSE and G. SHEARER; "A course in the solution of spherical triangles for the mathematical laboratory" by H. BELL; and "An introduction to the theory of automorphic functions" by L. R. FORD. The majority of the Edinburgh Tracts were inspired by the work of the mathematical laboratory in Edinburgh and are devoted to laboratory instruction, in the belief that "the student trained in modern pure mathematics is somewhat helpless in face of the concrete problems presented by the applied mathematical sciences" unless his training is supplemented by courses in "a mathematical laboratory, devoted to numerical and graphical calculation and analysis."

The fifteenth annual meeting of the Central Association of Science and Mathematics Teachers was held in Chicago on Friday and Saturday, November 26, 27. At the opening of the general session, Professor E. R. HEDRICK, of the University of Missouri, gave an address on "Required mathematics in secondary schools and colleges"; and EDWARD A. STEINER, professor of sociology at Grinnell College, Iowa, spoke on "A message from a sociologist to teachers of science and mathematics."

The mathematics section held two sessions at which committee reports were presented: on algebra by MARY JACKSON of Lincoln High School, Nebraska; on geometry by E. R. BRESLICH of the University of Chicago High School; on correlation of secondary mathematics by EDNA ALLEN of Iowa State Teachers College; and on vocational mathematics by KENNETH G. SMITH of Iowa State Agricultural College. There were also two addresses at the section meetings; one by Professor G. W. MYERS of the University of Chicago on "Current educational movements and general mathematics," and the other by Professor L. C. KARPINSKI of the University of Michigan on "The history of algebra, with stereopticon views." There was also a round table discussion on "History of

mathematics—its place and value in secondary mathematics," led by CHRISTINE BEDNAR, of Parker High School, Chicago, and followed by Professor KARPINSKI and others.

The call for an organization meeting of a new National Mathematical Association was signed by about four hundred and fifty persons representing every state in the Union, the District of Columbia, and Canada, and including high school, normal school, college, and university teachers, consulting engineers, actuaries, and others who are interested in mathematics purely for its own sake. The meeting was scheduled for Thursday, December 30, at ten o'clock A. M., in Page Hall, Ohio State University. Doubtless other sessions will be needed to complete the organization. Only one program was arranged, aside from the business sessions; namely, on Friday morning at nine o'clock, Professor L. C. Karpinski, of the University of Michigan, is to give an illustrated address on "The Story of Algebra," to which all mathematicians in attendance upon the Columbus meetings are invited. A full report of the organization of the new Association will be printed in the January issue of the MONTHLY, and a copy will be sent to all persons who signed the call and to all who request the same before the date of the meeting. Announcement of conditions of membership will be sent to all who have manifested interest in this movement, and the announcement stated that a proposition would be made at Columbus to admit to charter membership all who may join the new Association before April 1, 1916.

The MONTHLY closes its third year since its reorganization with many of its earlier fears and anxieties removed, with its subscription list more than trebled, with a large and increasing body of contributors, and with a feeling of certainty that no mistake was made in regard to the opportunity that seemed to be offered in the field in which its operations have been centered. High praise is due to those colleges and universities which deemed the cause represented by the MONTHLY to be worthy of their consideration and which made possible the promotion of this cause by their generous subsidy contributions throughout the three-year period. Hearty thanks are due the representatives of these institutions who have constituted the Board of Editors and who have unselfishly given their time and their services, amounting in the aggregate to no small consideration, thus showing their faith in this cause and their willingness to back it by works. Full credit is due the constituency of the MONTHLY, a widely representative body of persons who have shown their confidence and indicated their loyalty by their continued adherence in ever-increasing numbers. Finally, and not least, credit is due those who have made their contributions to the cause by purchasing advertising space, and to The New Era Printing Company, whose excellent composition and presswork are worthy of mention.

We believe that we can confidently promise for 1916 the continuation of all the worthy features in the make-up of the MONTHLY, the elimination of many of its faults, and the addition of several things which will increase its interest and broaden its influence.

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ERRATA IN VOLUME XXII.

Page 61, line 4 down, "$c = 1$" should read $c = 2$.

Page 68, line 13 down, "$f''(x) = -c(2n-1)\pi$" should read $f''(x) = e^{-c(2n-1)}\pi$.

Page 130, line 6 down, problem "430" should read 430A.

Page 131, line 4 down, "$\pi^3/12^3$" should read $\pi^3/12$.

Page 137, line 6 down, "$3a^2 + 2ab = A$" should read $3a^2 = 2ab = A$. Line 14 down, "dy/ax" should read dy/dx.

Page 161, line 14 down, problem "463" should read 463A.

Page 167, in equation (5), "DF/EF" should read DE/EF. Line 6 up, the first radical sign should extend over the numerator only.

Page 205, line 17 up, "log 12/log 2" should read log 12/log 6. Line 12 up, "Irvin" should read Irwin.

Page 251, line 3 up, the first word should be "or."

Page 292, line 18 down, "Cantorion" should read Cantorian.

Page 293, line 17 up, "creatrive" should read creative.

Page 294, line 1 down, "dissusions" should read discussions.

Page 319, line 11 down, "$\frac{108}{169}$" should read $\frac{208}{169}$.

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